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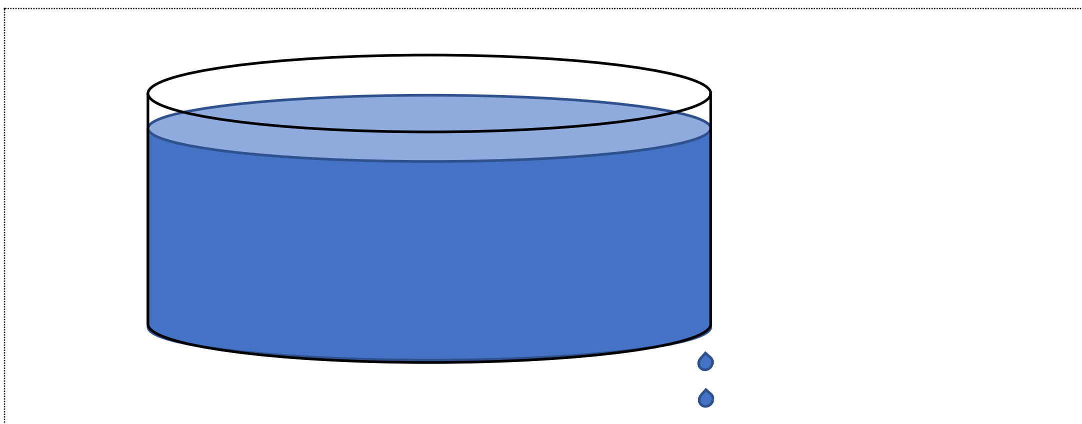
Plotting non-Linear Functions

Discovering Constants

In the pipe and bucket studies above, we were lucky to have functions that took the shape of a straight line. But what if we were studying the leak rate from a water tank, instead of from a pipe?

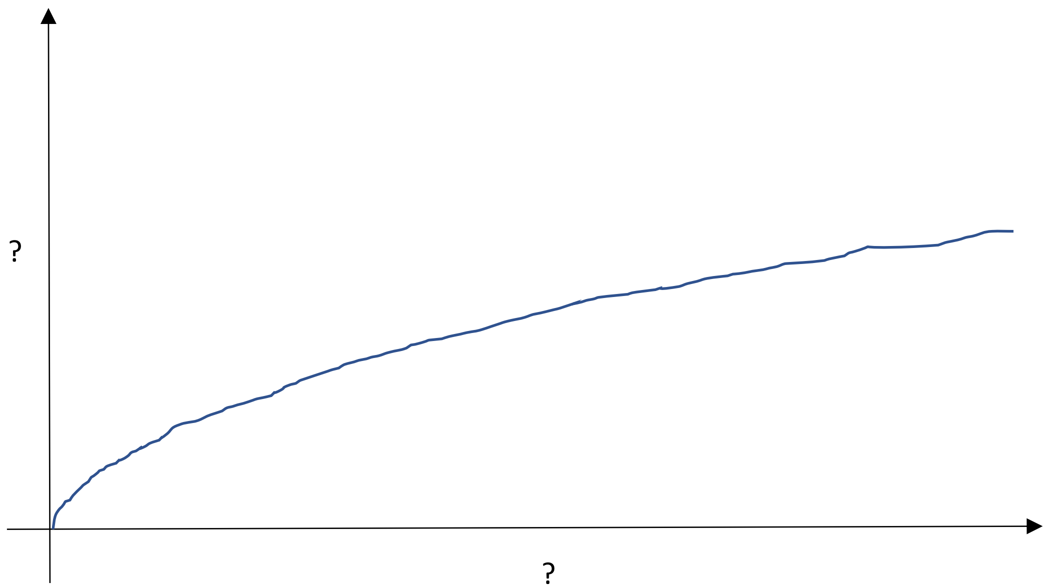


The leak rate from a vessel, for example a pipe, a tank, or even an artery or vein, depends on the pressure in that vessel, and the size and shape of the hole the leak is coming out of. In our pipe examples, we were assuming the hole in the pipe didn't change, and the pressure in the pipe didn't change.

In an open vessel like a tank, the pressure in the vessel depends on the depth where you are measuring it – the more meters of water (or any fluid) there are above the measurement point, the higher the pressure will be at the measurement point (which is why you may feel pressure in your ears if you swim to the bottom of a swimming pool). How will this affect flow rate?

The mathematical formula that relates flow rate to depth in an open vessel is called Torricelli's Law. It is $v = \phi\sqrt{2gh}$, where v is flow rate, ϕ is a constant that represents three other constants (describing the density of the liquid, and the size and shape of the hole), g is gravity, and h is the depth (or, rather, the height) of the liquid above the measurement point. (The law is named after Evangelista Torricelli, an Italian physicist and mathematician who lived in the 17^{th} Century. Torricelli invented the barometer).

The plot of the function that represents Torricelli's law would take the shape of a square root function, where v is the y-axis, and h is the x-axis. The plot would be scaled by g (gravity) and ϕ, but we don't know what ϕ is.

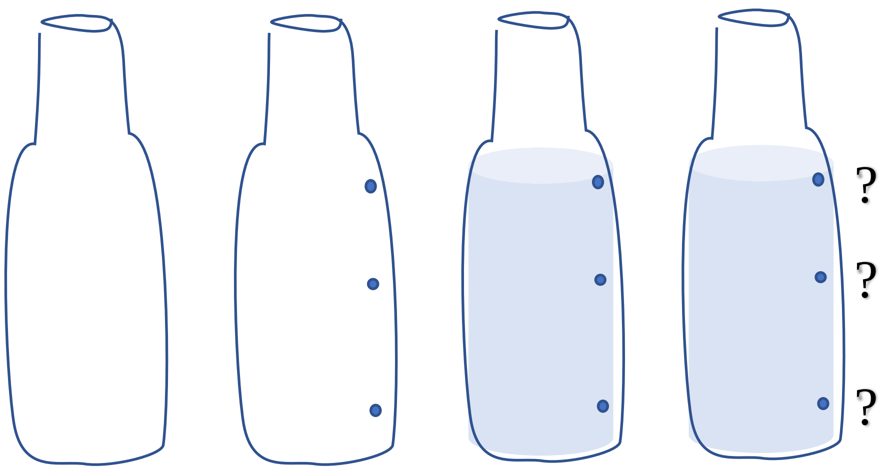


Figuring out ϕ by examining the hole the leak is coming out of, and by measuring the density of the fluid, would be very hard. But since the value of ϕ doesn't vary as the depth varies, even though the flow rate does, it isn't necessary to examine the hole, or to measure the density of the fluid – all an engineer has to do is to measure the flow rate out of the hole at one point in time, and the height of the water above the hole at that same point in time. The engineer can then put those numbers into the equation of Torricelli's Law to figure out a rough value for ϕ. (Of course, if the engineer wanted to be more accurate, they would measure several different pairs of flow rate and height, at several different times, as the level of the water goes down, and average their resulting values for ϕ). Once the engineer has figured out ϕ, the engineer can predict what the flow rate will be at any depth. From this range of flow rates (and, as long as the water in the tank doesn't freeze), the engineer can figure out (using Calculus) what the volume of the leak will be at any point in time.

Torricelli's law is hard to solve, but easy to demonstrate. All you need is an empty plastic water bottle and some water.

In this experiment we will use several holes at different heights, instead of just one hole as the water level drops in the vessel. But, you should still pay attention to how the flow rate from the bottom hole changes as the water level drops.

1 - Get an empty water bottle, and some water. You'll also want something to catch the water as it comes out of the bottle.
2 - Punch three small holes in the bottle, one at the very bottom, one near the middle, and one right below the neck. Try to make the holes as close to the same size and shape as possible.
3 - Have someone else block the holes with their fingers, and then fill up the bottle.
4 – Holding the bottle straight upright, unblock the holes.
What does the flow rate from the holes tell you about pressure in the bottle?

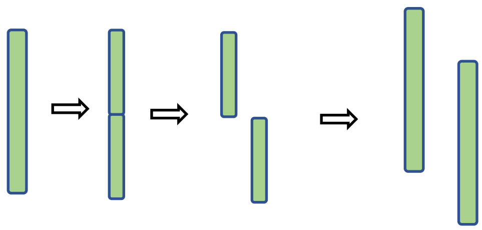


Exponents and Logarithms

As we've seen, not all things in the world work in straight lines. To model different kinds of systems, scientists have to use different kinds of equations, with different kinds of plots and charts. In the remainder of this section, we'll talk about some of those different kinds of equations.

One common function that appears in scientific research is called "exponential." Let's look back at the cholera epidemic in London. Epidemics are caused by microbes (that is, "germs"). During the Broad Street epidemic, microbes from one baby's diapers killed 500 people. How is this possible?

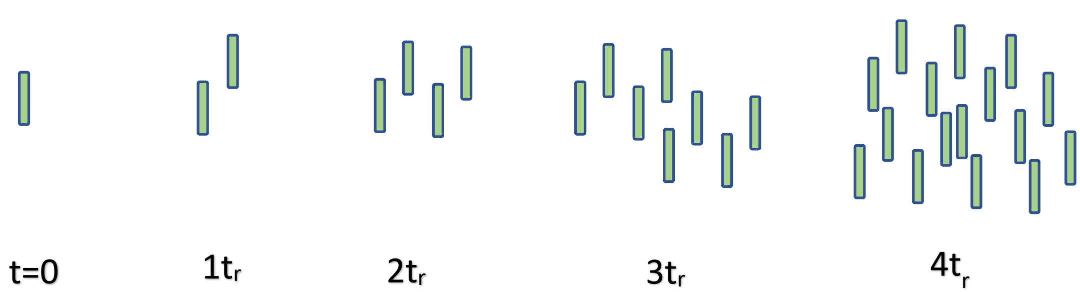
Cholera is caused by a specific kind of microbe, namely a strain of bacteria called "Vibrio Cholerae." Like most bacteria, Vibrio Cholerae reproduces by dividing itself in half.



When one bacterium splits, the two halves grow to be the same size as the original, and then those two halves also split (once the two halves have grown to be the same size as the original bacterium they are able to reproduce, because they are now adult bacteria). The time between the first split and the second is called a "reproduction cycle."

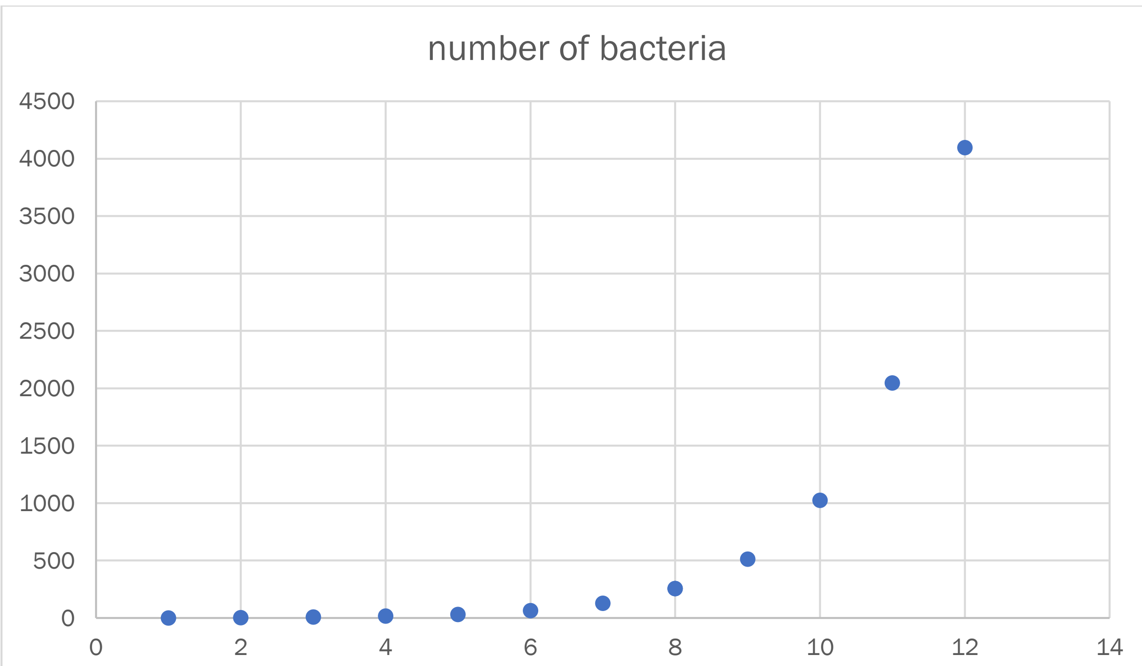
No matter how many bacteria you start with, if the bacteria reproduce by dividing in half the number of bacteria will double with every reproduction cycle. Each individual bacterium in a sample of bacteria won't divide at exactly the same time, but if you count the number of bacteria once every reproduction cycle, you can expect your value to always be about two times the previous value. Different strains and types of bacteria reproduce at different speeds, so the actual length in time of a reproduction cycle is different depending on the strain and type.

For simplicity, let's use "reproduction cycle" as our unit of time, instead of a specific number of minutes or hours. Let's call the time it takes a bacterium to reproduce, that is, the time it takes for it to go through a full reproduction cycle, "t_r". If at t=0 there is one bacterium, at $1*t_r$ there will be two bacteria. At $2* t_r$ there will be four bacteria. At $3* t_r$ there will be eight bacteria, and so on.



time (in t_r)	0	1	2	3	4	5	6	7	8	9	10	11
number of bacteria	1	2	4	8	16	32	64	128	256	1024	2048	4096

If we were to draw a graph of "number of Vibrio Cholerae" versus "time," that graph would have an upward slope, like the slope in the flow rates studies. Unlike the flow rate studies, however, the graph would look like $y=2^x$, or, rather, bacteria=2^{time}. The variable "time", which is our independent variable, appears as an exponent in this equation.



While the base in the above graph is 2, the general equation for an exponential is $y = b^x$ (or $y=b^x+ c$, where "c" is a constant). The base, b, in an exponential function can be any positive number. If bacteria reproduced by splitting into thirds, for example, the graph above would look like $y=3^x$, if bacteria reproduced by splitting into fourths, it would look like $y=4^x$, and so on. In all of these cases, however, the value of y, that is, the number of bacteria we are representing (or anything else being counted) would grow very fast.

For some kinds of graphing, y values grow so fast that they cannot be reasonably represented on an x-y plot. Below is a table of $y=10^x$ as x goes from 0 to 8. If a piece of graph paper holds 60 squares in the vertical direction, how many pieces of paper would it take to draw this graph as x goes from zero to 10?

x	0	1	2	3	4	5	6	7	8
10^x	1	10	100	1,000	10,000	100,000	1,000,000	10,000,000	100,000,000

How can we possibly plot these huge numbers? We can plot them using logarithms.

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